*Int. J. Heat Mass Transfer.* Vol. 14, pp. 1–14. Pergamon Press 1971. Printed in Great Britain

# SOME PROBLEMS OF FREE CONVECTION IN A MACROCAPILLARY

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(*Received* 2 *July* 1968)

**Аннотация**—Приводится решение ряда задач естественной конвекции в несжимаемой вязкой жидкости в элементарных макрокапиллярах при неоднородном распределении температуры на границе. Подробно изучена структура течения жидкости и влияние магнитного поля на конвекцию в случае проводящей жидкости. Дается оценка влияния свойств макрокапилляра на структуру течения, величину конвективной скорости и распределение температуры.

## NOMENCLATURE

| | |
|---|---|
| $l$, | length of molecular mean free path; |
| $h$, | radius of capillary; |
| $u, v$, | velocity components; |
| $\psi$, | stream function; |
| $p$, | pressure; |
| $\rho$, | density; |
| $T$, | temperature; |
| $T_w$, | temperature of walls; |
| $\nu$, | kinematic viscosity; |
| $\varkappa$, | thermal diffusivity; |
| $\bar{c}$, | sound velocity in gas; |
| $k$, | wave number; |
| $\varepsilon$, | temperature gradient along axis $y$; |
| $Pr$, | Prandtl number; |
| $M$, | Hartman number; |
| $P_m$, | magnetic Reynolds number; |
| $Gr$, | Grashof number; |
| $H$, | magnetic field; |
| $H_z, H_x, h_y$, | components of variable portion of magnetic field. |

THE STUDY of free convection in capillary-porous bodies under nonisothermal conditions is of great interest. Transfer phenomena in capillary-porous bodies are characterized by some peculiarities connected with the value of the ratio of the mean free molecular path $l$ to the capillary radius $h$ [1]. If the capillary radius $h$ is of the order of the mean free molecular path $1(h \sim 10^{-7}$ m for air), then the transfer laws depend on free-molecular conditions and should be caculated using kinetic concepts. This is the case of a highly rarefied gas. If the value $l/h$ is in comparison to unity (weakly rarefied gas), heat transfer processes can be treated in a macroscopic way. In this representation the usual statement of the problem should be somewhat altered. In the equations of motion and of state the correction terms are extremely small and will be neglected in further considerations [1]. However the boundary conditions on the solid surface require fundamental consideration. The derivation of new macroscopic conditions is carried out with the help of kinetic representation [3].

In hydrodynamics the conditions of adhesion are usually assumed, i.e. it is considered that the velocity and temperature of the gas at the wall are equal to the velocity and temperature of the wall. New boundary conditions allow for jumps in the tangential components of the velocity and temperature which are proportional to normal derivatives of these values in the Navier–Stokes approximation.

In the present paper the problem is formulated and solved on natural convection in a macrocapillary at $l/h < 1$ and nonuniform temperature distribution at the boundary. Particular attention is paid to elucidation of the effect of slip conditions and temperature jump on the flow structure in a macropore. One of the simplest capillary models, horizontal capillary slits with geometric dimension $h$, is considered.

The Cartesian system of coordinates $x$, $y$ is employed. The walls coincide with the planes $y = 0$ and $y = h$. The gravity field is in the positive direction of the axis $y$. Let $u$, $v$ be the velocity components along the axes $x$, $y$; $p$, $\rho$, $T$ are pressure, density and temperature; the kinematic viscosity $v$ and thermal diffusivity are assumed constant.

The initial system of equations is of the form

$$\left.\begin{aligned}
&\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\\
&\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho g\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\\
&u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \varkappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)\\
&f(p, \rho, T) = 0.
\end{aligned}\right\} \tag{1}$$

The boundary conditions, which take into account the temperature and velocity jumps are formulated as follows

$$\left.\begin{aligned}
&y = 0, h: u = 2\pi n_1\frac{l}{L}\frac{\partial u}{\partial y} + 2\pi n_2\frac{l}{L}\bar{c}\frac{\partial T}{\partial x}, v = 0\\
&y = 0 \quad T - T_w = 2\pi n_3\frac{l}{L}\frac{\partial T}{\partial y}, \qquad T_{w_0} = T_0(1 + a\sin kx)\\
&y = h \quad T - T_w = 2\pi n_3\frac{l}{L}\frac{\partial T}{\partial y}, \qquad T_{w_h} = T_h(x)\\
&n_1 = \tfrac{5}{16}\pi L \quad n_2 = \frac{15}{128}\pi\frac{L}{T_0} \quad n_3 = \tfrac{75}{128}\pi L
\end{aligned}\right\} \tag{2}$$

where $\bar{c}$ is a characteristic velocity usually assumed equal to the sound velocity. The case will be considered of one of the walls being kept at constant temperature, and the temperature of the other wall varying periodically

$$T_{w_0} = T_0(1 + a \sin kx), T_{w_h} = T_h = \text{const.} \tag{3}$$

It is known that in uniform heating from below there exists a critical temperature drop $\Delta T_{cr}$. When this value is exceeded [4], mechanical equilibrium is disturbed. The conditions of heat conduction are changed to free convection. In the case of non-uniform temperature distribution at the boundary as shown in [2], there is no limiting value of the temperature difference, i.e. convection arises at any small nonuniformities of the temperature. If the maximum temperature difference $T_{w_h} - T_{w_0}$ exceeds $\Delta T_{cr}$, the flow structure in a capillary is rather complicated. In the present case the study will be confined to the condition when $T_{w_h} - T_{w_0} < \Delta T_{crit}$ and the amplitude of periodic variation $a$ is small. This allows a linearized solution of the problem. The solution of the system of equations (1)–(3) is of the form

$$\left.\begin{aligned} &T = T_0(1 + T) \qquad p = p_0(1 + p) \qquad \rho = \rho_0(1 + \rho) \\ &u = 0 + u \qquad\quad\; v = 0 + v \end{aligned}\right\} \tag{4}$$

where $T$, $p$, $\rho$, $u$, $v$ are deviations from equilibrium values.

The linearized basic set of equations is of the form

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \qquad \frac{\partial P}{\partial x} = \mu_* \Delta u; \qquad \frac{\partial P}{\partial y} = \mu_* \Delta v - \beta_* T \\ &\varepsilon v + \varkappa \Delta T = 0 \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} &x = 2\pi x/L; \qquad y = 2\pi y/L; \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; \\ &\mu_* = \frac{2\pi\mu}{P_0 L}; \qquad \beta_* = \frac{gL\beta T_0}{P_0 2\pi}; \qquad k = \frac{2\pi}{L}. \end{aligned}\right\} \tag{6}$$

The boundary conditions remain the same.

The solution of the problem stated is

$$v = (A_3 Z_1 + A_0 Z_2 - A_2 Z_3 - A_5 Z_4 + A_1 Z_5 + A_1 Z_5 + A_4 Z_6) \sin kx \tag{7}$$

$$u = \left\{ -\frac{\lambda_1}{k} A_0 Z_1 - \frac{\lambda_1}{k} A_3 Z_2 + \frac{\lambda_0}{k}\left(A_1 \sin\frac{\theta}{2} - A_5 \cos\frac{\theta}{2}\right) Z_3 + \frac{\lambda_0}{k}\left(A_4 \sin\frac{\theta}{2} - A_2 \cos\frac{\theta}{2}\right) Z_4 \right.$$
$$\left. - \frac{\lambda_0}{k}\left(A_4 \cos\frac{\theta}{2} + A_2 \sin\frac{\theta}{2}\right) Z_5 - \frac{\lambda_0}{k}\left(A_1 \cos\frac{\theta}{2} + A_5 \sin\frac{\theta}{2}\right) Z_6 \right\} \cos kx \tag{8}$$

$$T = \left\{ \frac{\mu}{\beta k^2} [S A_3 Z_1 + S A_0 Z_2 - (\lambda_0^2 F A_4 + H A_2) Z_3 - (\lambda_0^2 F A_1 + H A_5) Z_4 \right.$$
$$\left. + (H A_1 + \lambda_0^2 F A_5) Z_5 + (H A_4 + \lambda_0^2 F A_2) Z_6] \right\} \sin kx \tag{9}$$

where

$$\left.\begin{aligned}
&Z_1 = \sinh \xi_1 \qquad Z_2 = \cosh \xi_1 \qquad Z_3 = \cosh \eta_2 \sin \eta_1\\
&Z_4 = \sinh \eta_2 \sin \eta_1 \qquad Z_5 = \cosh \eta_2 \cos \eta_1 \qquad Z_6 = \sinh \eta_2 \cos \eta_1\\
&\xi_1 = \lambda_1 y \qquad \eta_1 = \lambda_0 \sin \frac{\theta}{2} y \qquad \eta_2 = \lambda_0 \cos \frac{\theta}{2} y\\
&\theta = \tan^{-1} \frac{\sqrt{3}}{2[k^2h^2 + (\delta/2)]} \qquad \delta = h^2 \left(\frac{\varepsilon\beta k^2}{\mu\varkappa}\right)^{\frac{1}{2}}\\
&F = -\lambda_0^2 \sin 2\theta + 2k^2 \sin\theta, \qquad \lambda_0^2 = k^2h^2 - h^2\left(\frac{\varepsilon\beta k}{\mu\varkappa}\right)^{\frac{1}{2}}\\
&H = \lambda_0^4 \cos 2\theta + 2k^2\lambda_0^2 \cos\theta + k^4, \qquad S = \lambda_1^4 - 2k^2\lambda_1^2 + k^4\\
&\lambda_1 = \left\{\left[\frac{h^2k^2}{2} - \frac{h^2}{2}\left(\frac{\varepsilon\beta k^2}{\mu\varkappa}\right)^{\frac{1}{2}}\right]^2 + \left[\frac{\sqrt{3}}{2}k^2h^2 - \frac{\sqrt{3}}{2}h^2\left(\frac{\varepsilon\beta k^2}{\mu\varkappa}\right)^{\frac{1}{2}}\right]^2\right\}^{\frac{1}{2}}.
\end{aligned}\right\} \tag{10}$$

For easy analysis of the flow pattern the expression for the stream function is used

$$\psi = -\frac{1}{k}(A_3Z_1 + A_0Z_2 - A_2Z_3 - A_5Z_4 + A_1Z_5 + A_4Z_6)\cos kx. \tag{11}$$

The equations obtained are very bulky. To make the effects of temperature and velocity jumps more obvious, consider the limiting case $kh \ll 1$ and $kh$ to $\gg 1$. Let $kh \ll 1$, then equations (7)–(11) can be simplified to

$$v = \left(\frac{c_1}{120}y^5 + \frac{c_2}{24}y^4 + \frac{c_3}{6}y^3 + \frac{c_4}{2}y^2 + C_5y + c_6\right)\sin kx \tag{12}$$

$$u = \frac{1}{k}\left(\frac{c_1}{24}y^4 + \frac{c_2}{6}y^3 + \frac{c_3}{2}y^2 + c_4y + c_5\right)\cos kx \tag{13}$$

$$T = \frac{\mu}{\beta}\left[\frac{c_1k^2}{120}y^5 + \frac{c_2k^2}{24}y^4 + \left(\frac{c_3k^2}{6} - \frac{c_1}{3}\right)y^3 + \left(\frac{c_4k^2}{2} - c_2\right)y^2 + \left(\frac{c_1}{k^2} - 2c_3 + c_5k^2\right)y + \frac{c_2}{k^2} - 2c_4 + c_6\right]\sin kx \tag{14}$$

$$\psi = -\frac{1}{k}\left(\frac{c_1}{120}y^5 + \frac{c_2}{24}y^4 + \frac{c_3}{6}y^3 + \frac{c_4}{2}y^2 + c_5y + c_6\right)\cos kx. \tag{15}$$

But integration constants $c_i$ continue to be bulky and are thus not presented here. It should be noted that in this limiting case free convection practically does not affect the flow pattern in a pore. The motion is completely attributed to the slip and jump of temperature. Now let $\varepsilon = 0$, $kh \gg 1$, then the solution is of the form

$$u = [\alpha y^2 + \gamma y - (\gamma + 2\alpha)]\,e^{-ky}\cos kx \tag{16}$$

$$v = \left[-\frac{\alpha y^2}{k} + (\gamma + 2\alpha)y\right]e^{-ky}\sin kx \tag{17}$$

$$T = b\,e^{-ky}\sin kx \qquad b = a[1 + 2\pi n_3(l/L)]^{-1} \tag{18}$$

$$\Psi = -\frac{1}{k}\left[-\frac{\alpha y^2}{k} - (\gamma + 2\alpha)y\right]e^{-\kappa y}\cos kx$$

$$\alpha = \frac{\beta_* b}{8\mu_*} \qquad \gamma = \frac{-2\alpha[1 + 2\pi n_1(l/L)] + 2\pi n_2 \bar{c} b(l/L)}{1 + 4\pi n_1(l/L)}. \tag{19}$$

By letting the limit $l/h \to 0$ in (7)–(11) results are obtained corresponding to large gas pressures in a capillary. Here motion in the pore results only from the effect of the gravity field and the nonuniformity of the temperature distribution. In this case the nature of the flow and heat transfer are in

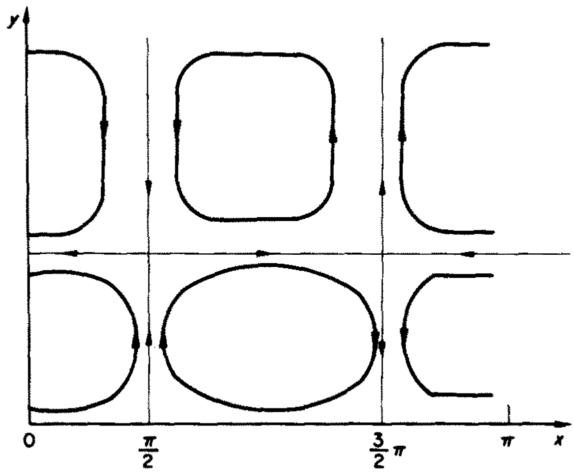


FIG. 1. Flow pattern in a macrocapillary $kh \gg 1$, $\varepsilon = 0$.

agreement with free convection in an ordinary dense gas. In another limiting case when the gravity field is absent ($g = 0$) the flow also exists but only when $l/h \neq 0$, i.e. at low pressures and rather small pores. It appears that the character of the periodicity along the axis $x$ depends neither on pore sizes and pressure in a capillary, nor on the presence of the gravity field and is attributed only to the nonuniformity of the temperature of the pore walls. It is interesting to note that zero stream functions correspond to the position of extreme values of temperature.

Consider the case $kh \gg 1$ in detail. In a capillary the period of temperature variations is very small Thus, for example, for a macrocapillary with $h \sim 10^{-6}$ m, $k \gg 10^6$ m and $\lambda = 2\pi/k \ll 2\pi \cdot 10^{-6}$ m, i.e. it is the most important case when nonuniformity of temperature distribution over the capillary wall is seen. Let us dwell on an important feature of free convective motion in a macropore, i.e. the origin of two flow regions (cells) along the coordinates $y$. From the expression for the stream function $\psi$ it follows that the position of the boundary between the two regions (zero stream line) is determined as follows

$$y = -\left(2 + \frac{\gamma}{\alpha}\right), \qquad \frac{\gamma}{\alpha} < 0 \qquad \left|\frac{\gamma}{\alpha}\right| > 2. \tag{20}$$

It is not difficult to explain which factor, the gravity field or temperature and velocity jumps, affect the flow in the various regions. As seen from (20) with weakening gravity field, the boundary between the regions is displaced towards large values of $y$, i.e. the lower near-wall cell occupies the whole capillary volume. Therefore the lower near-wall flow (primary cell) is first of all due to the conditions of temperature and velocity jumps, while the flow in the upper cell mainly to free convection. The minimum value of the gravity field when, at given range of temperature variation and for a given liquid, the flow in a pore is divided into two regions is given by

$$g_{\min} = \frac{-8\mu\gamma k^2}{(kh + 2)\beta T_0 b}; \qquad \gamma < 0. \tag{21}$$

With an increase in the gravity field the size of the primary cell decreases and tends to zero. The size of the primary cell can also be diminished by increasing the amplitude of temperature variations $b$. However at large values of $b$, firstly the authors' expansions may not be justified and, secondly, the laminar nature of the flow can be disturbed. In the later case there exists a critical size of both cells for laminar motion. The results obtained make it possible to calculate the effective size (from the viewpoint of motion intensity) of a macrocapillary. The expression for the effective size of the macrocapillary when $|\gamma/\alpha| \gg 2$ is

$$h = 2\pi l\sqrt{(n, n_3)}$$

For air at normal conditions $h \sim 10^{-6}$ m. As the estimations of the velocity show, in such a capillary the motion is basically attributed to the effects of slip and temperature jump. Contribution to the velocity due to free convection is of similar order of magnitude at $1/h \sim 10^{-3}$ m. The velocity is of the order $10^{-4}$–$10^{-5}$ m/s if the amplitude of temperature oscillations is of the order of one degree.

Consider the problem of free convection in a horizontal macrocapillary filled with electroconducting fluid. The vector of permanent magnetic field intensity is parallel to the gravity field. The dimensionless form of the basic linarized equations is as follows [5]

$$\frac{\partial}{\partial x}\left(p + \frac{M^2}{P_m}\frac{H^2}{2}\right) = \nabla^2 u + \frac{M^2}{P_m}\frac{\partial h_x}{\partial y} \tag{22}$$

$$\frac{\partial}{\partial y}\left(p + \frac{M^2}{P_m}\frac{H^2}{2}\right) = \nabla^2 v - GrT + \frac{M^2}{P_m}\frac{\partial h_y}{\partial y} \tag{23}$$

$$\frac{1}{P_m}\nabla^2 h_x = -\frac{\partial u}{\partial y} \tag{24}$$

$$\frac{1}{P_m}\nabla^2 h_y = -\frac{\partial u}{\partial x} \tag{25}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{26}$$

$$\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} = 0 \tag{27}$$

$$\frac{1}{Pr}\nabla^2 T = -\varepsilon v \tag{28}$$

where

$$Pr = \nu/\varkappa \qquad M = (B_0 h/c)\sqrt{(\delta/\mu)} \qquad P_m = \nu/\lambda_2\lambda = C^2/4\pi\sigma.$$

The boundary conditions for the temperature and velocity remain the same, equations (2). The tangential component of the magnetic field at the boundary is continuous.

It has already been mentioned that the effect of free convection is most important at $kh \gg 1$. This is the case to be considered. Assume that there is no temperature gradient across the macropillary $\varepsilon = 0$, which allows solution of the energy equation (28) independently of other equations in the system (22)–(27). Without dwelling on computational details the solution of the system of equations is

$$u = \left[A\,e^{-\varkappa_1 y} + B\,e^{-\varkappa_2 y} + \frac{Grb}{M^2}e^{-ky}\right]\cos kx$$

$$v = -\left[\frac{Ak}{\kappa_1}e^{-\varkappa_1 y} + \frac{Bk}{\varkappa_2}e^{-\varkappa_2 y} + \frac{Grb}{M^2}e^{-ky}\right]\sin kx$$

$$\psi = \left[\frac{A}{\varkappa_1}e^{-\varkappa_1 y} + \frac{B}{\varkappa_2}e^{-\varkappa_2 y} + \frac{Grb}{M^2 k}e^{-ky}\right]\cos kx$$

$$h_x = \left\{C\,e^{-ky} - P_m\left[\frac{A\varkappa_2}{\varkappa_1^2 - k^2}e^{-\varkappa_1 y} + \frac{B\varkappa_2}{\varkappa_2^2 - k^2}e^{-\varkappa_2 y} - \frac{Grb}{2M^2}(1-y)\,e^{-ky}\right]\right\}\cos kx$$

$$h_y = \left\{C\,e^{-ky} - kP_m\left[\frac{A}{\varkappa_1^2 - k^2}e^{-\varkappa_1 y} + \frac{B}{\varkappa_2^2 - k^2}e^{-\varkappa_2 y} - \frac{Grb}{2kM^2}y\,e^{-ky}\right]\right\}\sin kx$$

$$T = b\,e^{-ky}\cos kx$$

$$\varkappa_1 = \sqrt{\{\tfrac{1}{2}(2k^2 + M^2) + \sqrt{[\tfrac{1}{4}(2k^2 + M^2)^2 - k^4]}\}}$$

$$\varkappa_2 = \sqrt{\{\tfrac{1}{2}(2k^2 + M^2) - \sqrt{[\tfrac{1}{4}(2k^2 + M^2)^2 - k^4]}\}}$$

where the coefficients $A$, $B$, $C$ are found from the boundary conditions

$$A = \frac{\delta_1}{\Delta} \qquad B = \frac{\delta_2}{\Delta} \qquad C = \frac{\delta_3}{\Delta}$$

$$\Delta = \frac{k}{\varkappa_1}\left(1 + 2\pi n_1 \frac{l}{L}\varkappa_2\right) - \frac{k}{\varkappa_2}\left(1 + 2\pi n_1 \frac{l}{L}\varkappa_1\right)$$

$$\delta_1 = -\frac{k}{\varkappa_2}\left(-\frac{Grb}{M^2} + 2\pi n_1 \frac{l}{L}\frac{Grbk}{M^2} - 2\pi n_2 \bar{c}\frac{l}{L}bk\right) - \frac{Grb}{M^2}\left(1 + 2\pi n_1 \frac{l}{L}\varkappa_2\right)$$

$$\delta_2 = \frac{Grb}{M^2}\left(1 + 2\pi n_1 \frac{l}{L}\varkappa_1\right) + \left(-\frac{Grb}{M^2} + 2\pi n_1 \frac{l}{L}\frac{Grbk}{M^2} - 2\pi n_2 \bar{c}\frac{l}{L}bk\right)\frac{k}{\varkappa_1}$$

$$\delta_3 = \begin{vmatrix} 1 + 2\pi n_1 \frac{l}{L}\varkappa_1, & 1 + 2\pi n_1 \frac{l}{L}\varkappa_2, & -\frac{Grb}{M^2} + 2\pi n_1 \frac{l}{L}\frac{Grbk}{M^2} - 2\pi n_1 \frac{l}{L}\bar{c}bk, \\ \frac{k}{\varkappa_1}, & \frac{k}{\varkappa_2}, & -\frac{Grb}{M^2}, \\ \frac{kP_m}{\varkappa_1^2 - k^2}, & \frac{kP_m}{\varkappa_2^2 - k^2}, & 0 \end{vmatrix}$$

Consider the magnetic field effect on the flow structure in a macrocapillary in two limited cases of small and large Hartman numbers $M$.

For small $M$ within $O(M^2)$ the velocity components and the stream functions may be presented in the form

$$u = u_0 + M^2 u_1 \qquad v = v_0 + M^2 v_1 \qquad \psi = \psi_0 + M^2 \psi_1$$

$$u - u_0 = M^2 \left\{ \frac{\alpha k^2}{12} y^4 - \tfrac{1}{6}(4\alpha k^2 + 2\alpha k + k^2\gamma) y^3 + \left[ \alpha k^2 + k(2\alpha + k\gamma) + \frac{\gamma}{2}(2k+1) \right] y^2 \right.$$
$$\left. + [2\gamma(2k+1) - k(\alpha + k\gamma)] y - \tfrac{1}{2}\alpha - 3k\gamma + \frac{5\gamma}{8k} - \frac{\alpha}{16k} + \frac{\alpha}{4k^2} \right\} e^{-ky} \cos kx.$$

As was expected the sign of the velocity variation under the action of a magnetic field is negative, i.e. the velocity decreases with an increasing magnetic field. For the case of small $M$ the displacement

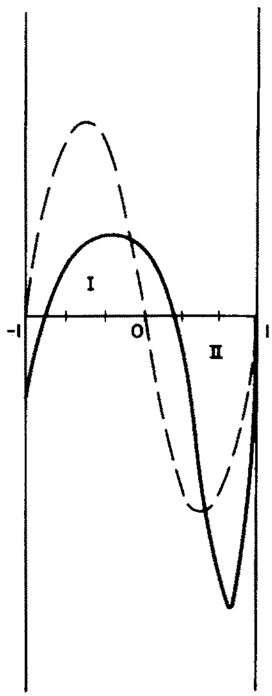


FIG. 2. ———— velocity profile in a vertical macrocapillary; – – – – – velocity profile in a vertical channel.

of the boundary between two flow regions due to the magnetic field is obtained when the stream function is equal to zero. From the expression

$$u_1 = -\frac{\partial \psi_1}{\partial y}$$

it is seen that since the sign of the term including $M^2$ is negative, the boundary between the cells is displaced towards the wall.

At large Hartman numbers a noticeable redistribution of the velocity components due to Archimedean forces and slip conditions (2) takes place. It is interesting to note that free-convective motion may be depressed with increase of the Hartman number, while the "proper" motion due to the slip conditions cannot.

So far horizontal macrocapillaries have been discussed. Now consider the peculiarities of the flow due to free convection in a vertical macrocapillary.

Let the macrocapillary walls be parallel to the gravity vector $\vec{g}$, i.e. the macrocapillary is placed vertically. The wall temperatures are kept constant and different. The capillary is placed in a permanent magnetic field $H_0$ normal to the walls ($H_x = H_0$). Assume that all the values depend only on the coordinate $X$. This agrees with real conditions in the capillary whose width $h$ is much less then the height $L$. The losses due to the Joule heat are neglected. The initial one-dimensional system is written as follows

$$\frac{\partial}{\partial x}\left(p + \frac{M^2}{P_m}\frac{H^2}{2}\right) = 0 \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(p + \frac{M^2}{P_m}\frac{H^2}{2}\right) = \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - GrT + \frac{M^2}{P_m}\frac{\mathrm{d}H_z}{\mathrm{d}x} \tag{30}$$

$$\frac{1}{P_m}\frac{\mathrm{d}^2 H_z}{\mathrm{d}x^2} = -\frac{\mathrm{d}v}{\mathrm{d}x} \tag{31}$$

$$\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} = 0. \tag{32}$$

In this case the boundary conditions (2) reduce to the following

$$\begin{aligned} x = \pm 1 \qquad & v = 2\pi n_1 \frac{l}{h}\frac{\partial v}{\partial x} + 2\pi n_2 \frac{l}{h}\bar{c}\frac{\partial T}{\partial x} \\ & T - T_w = 2\pi n_3 \frac{l}{h}\frac{\partial T}{\partial x} \qquad H_z = 0 \end{aligned} \tag{33}$$

where

$$n_1 = \tfrac{5}{16}\pi \qquad n_2 = \tfrac{15}{128}\pi \qquad n_3 = \tfrac{75}{128}\pi.$$

From equation (32) it follows that temperature distribution across the capillary is linear, the transverse component of the magnetic field is $H_x = H_0$, and pressure $P$ may be a linear function of $z$. Simultaneous solution of equations (30) and (31) yields distribution of the velocity $v(x)$ and longitudinal component of the magnetic field $H_z(x)$

$$T = -x - 2\pi n_3 \frac{l}{h} \tag{34}$$

$$v = \frac{Gr}{M^2}\left(x + 2\pi n_1 \frac{l}{h}\right) - \frac{Gr}{M^2}\frac{\sinh Mx}{\sinh M} - \frac{Gr}{M\sinh M}\, 2\pi n_1 \frac{l}{h}\cosh Mx \tag{35}$$

$$H_z = \frac{GrP_m}{M^2}\left[\frac{\cosh Mx}{M\sinh M} - \frac{x^2}{2} + \frac{1}{2} - \frac{\cosh M}{M\sinh M} - 2\pi n_1 \frac{l}{h}x + 2\pi n_1 \frac{l}{h}\frac{\sinh Mx}{\sinh M}\right]. \tag{36}$$

Note that terms of the order $(1/h)^2$ included in the solution (35), (36) need not be taken into account since the initial system of equations and boundary conditions is written within the accuracy to $(l/h)$.

Let us discuss the results obtained. The effect of temperature and velocity jumps on the shape of the temperature and velocity profiles are ascertained first. From (34) it is clearly seen that the temperature gradient $\partial T/\partial x$ does not depend on the value of $l/h$, and the absolute value of the temperature increases. In a rarefied gas the shape of the velocity profile changes. Thus the coordinates of the points where the velocity is maximum are displaced in comparison with the case of a dense gas. If the wall is hot, they move closer to it, if it is cold, they move away from it. The value of the displacement

$$\Delta x_{\text{displ}} = x_{\max}(l/h = 0) - x_{\max}(l/h \neq 0) \tag{37}$$

is proportional to the value of $l/h$

$$x_{\max\,\mathrm{I}} = -\left(\frac{1}{\sqrt{3}} + 2\pi n_1 \frac{l}{h}\right) \qquad x_{\max\,\mathrm{II}} = \frac{1}{\sqrt{3}} - 2\pi n_1 \frac{l}{h}. \tag{38}$$

Due to the boundary conditions of slip at the walls the velocity $v$ is equal to $\frac{2}{3}\pi n_1(l/h)$.

Unlike the case of a dense gas ($l/h = 0$), at zero pressure gradient the flow rate would be different from zero. For keeping a zero flow rate, a negative constant pressure gradient should be maintained.

$$\frac{\mathrm{d}p}{\mathrm{d}z} = 2\pi \frac{l}{h} Gr(n_1 - n_3). \tag{39}$$

Note that the value of the vertical heat flux is independent of slip conditions

$$Q = c_p\rho \int_{-h}^{h} vT\,\mathrm{d}x. \tag{40}$$

Let us cite some numerical estimates. Thus at $Gr = 10^3$, $l/h = 0{\cdot}01$, velocities at the wall $v_w \sim 10^{-4}$ m/s, maximum velocity in the region I decreases by the value $\Delta V_{\mathrm{I}} \simeq 5$ per cent, and increases by $\Delta V_{\mathrm{II}} \sim 4$ per cent in the region II.

Now that the effect of temperature and velocity jumps is clarified, let us study the peculiarities of convection appearing in a rarefied electrically conducting gas in the presence of a magnetic field. Since the losses due to the Joule heat are neglected, the temperature profile remains linear as in the case without a magnetic field. The velocity profile, however, changes its shape. Thus, in the case of small Hartman numbers within $O(M^2)$ the velocity may be presented in the form $v_*(x)[1 - (M^2/6)]$† where $v_*(x)$ is the velocity distribution in the absence of a magnetic field. This means that the characteristic points (position of maximum and zero velocity) have the same coordinates, but the absolute values of the velocity decrease. At large $M$ numbers, as is seen from (35), the velocity decreases everywhere except in the near-wall region as $1/M^2$.

In the vicinity of the walls the suppression of convection by the magnetic field decreases due to the slip effect and is proportional only to $1/M$. The velocity at the wall is

$$v(x = \pm 1) = 2\pi n_1 \frac{l}{h}\frac{Gr}{M^2}(M \coth M - 1). \tag{41}$$

† $v_* = Gr\,\dfrac{-x^3 + x}{6} - \pi n_1 \dfrac{l}{h} x^2 + \dfrac{1}{3}\pi n_1 \dfrac{l}{h}$

It is of interest to note that the magnetic field displaces the zero stream line still further to the hot wall. The upflow region contracts. It is not difficult to ascertain to what position the maximum velocities are being displaced to. By making the velocity component equal to zero, the positions of the maximum at large $M$ are determined by the expressions

$$x_{\max \mathrm{I}} = -1 + \frac{1}{M}\ln\left(2\pi n_1 \frac{l}{h}\right)$$
$$x_{\max \mathrm{II}} = 1 - \frac{1}{M}\ln\left(2\pi n_1 \frac{l}{h}\right). \tag{42}$$

Contrary to the case of a dense gas, the distribution of the magnetic field intensity due to free convection, as seen from expression (36), becomes asymmetric because of the terms proportional to $l/h$.

$$H_z - H_z(l/h = 0) = 2\pi n_1 \frac{GrP_m}{M^2}\frac{l}{h}\left(x - \frac{\sinh Mx}{\sinh M}\right). \tag{43}$$

In small magnetic fields the $H_0$ component of the magnetic field $H_z$ is determined by the expression

$$H_z = \frac{GrP_m}{6}\left(1 - \frac{M^2}{6}\right)\left[\frac{1 - x^2}{2} - 2\pi n_1 \frac{l}{h}(x - x^3)\right]. \tag{44}$$

With large magnetic fields $H_0$, free convection is suppressed and the component $H_z$ decreases within the whole region as $1/M^2$. Note that the temperature jump and slip conditions do not affect the mean vertical convective heat flux $Q$

$$Q = c_p \rho \int_{-h}^{h} vT\,\mathrm{d}x.$$

This coincides with the expression for $Q_M$ obtained in [5].

The pressure gradient necessary for keeping the flow rate at zero does not depend on the presence of a magnetic field and remains the same (39).

Thus, in a macrocapillary, the free convection is characterized by some striking peculiarities. The velocity profile and distribution of the induced magnetic field component $H$ become asymmetric. In a macrocapillary the change in the absolute value of the velocity may reach 5 per cent in comparison with the velocity in an ordinary slit.

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## APPENDIX

$$A_0 = \frac{\delta_0}{\Delta} \qquad A_1 = \frac{\delta_1}{\Delta} \qquad A_2 = \frac{\delta_2}{\Delta} \qquad A_3 = \frac{\delta_3}{\Delta} \qquad A_4 = \frac{\delta_4}{\Delta} \qquad A_5 = \frac{\delta_5}{\Delta}$$

$$\Delta = \begin{vmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \beta_{21} & \beta_{22} & \beta_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \beta_{31} & \beta_{32} & \beta_{33} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \beta_{41} & \beta_{42} & \beta_{43} \\ \alpha_{51} & \alpha_{52} & \alpha_{53} & \beta_{51} & \beta_{52} & \beta_{53} \\ \alpha_{61} & \alpha_{62} & \alpha_{63} & \beta_{61} & \beta_{62} & \beta_{63} \end{vmatrix}$$

$\delta_i$ is obtained from $\Delta$, if $i$th column is substituted by

$$\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a \end{vmatrix}$$

$$\alpha_{11} = \alpha_{12} = 1$$

$$\alpha_{21} = -n_1 \frac{\lambda_1^2}{k} - n_2 \frac{\mu}{\beta k} S; \qquad \alpha_{22} = -n_1 \frac{\lambda_0^2}{k} \cos\theta - n_2 \frac{\mu}{\beta k} H;$$

$$\alpha_{23} = \frac{\lambda_0}{k} \sin\frac{\theta}{2}; \qquad \beta_{21} = \frac{\lambda_1}{k}; \qquad \beta_{22} = \frac{\lambda_0}{k} \cos\frac{\theta}{2};$$

$$\beta_{23} = -n_1 \frac{\lambda_0}{k} \sin\theta - n_2 \frac{\mu\lambda_0^2}{\beta k} F;$$

$$\alpha_{31} = S; \qquad \alpha_{32} = H; \qquad \alpha_{33} = n_3 \left(\lambda_0 \sin\frac{\theta}{2} H - \lambda_0^3 \cos\frac{\theta}{2} F\right);$$

$$\beta_{31} = -n_3 \lambda_1 S; \qquad \beta_{32} = -n_3 \lambda_0^3 \left(H \cos\frac{\theta}{2} - F \sin\frac{\theta}{2}\right); \qquad \beta_{33} = \lambda_0^2 F$$

$$\alpha_{41} = \cosh\lambda_1; \qquad \alpha_{42} = \cosh\left(\lambda_0 \cos\frac{\theta}{2}\right); \qquad \alpha_{43} = -\cosh\left(\lambda_0 \cos\frac{\theta}{2}\right) \sin\left(\lambda_0 \sin\frac{\theta}{2}\right)$$

$$\beta_{41} = \sinh\lambda_1; \qquad \beta_{42} = \sinh\left(\lambda_0 \cos\frac{\theta}{2}\right); \qquad \beta_{43} = -\sinh\left(\lambda_0 \cos\frac{\theta}{2}\right) \sin\left(\lambda_0 \sin\frac{\theta}{2}\right)$$

$$\alpha_{51} = \frac{\lambda_1}{k} \sinh\lambda_1 + n_1 \frac{\lambda_1^2}{k} \cosh\lambda_1 - n_2 \frac{\mu}{\beta k} S \cosh\lambda_1$$

$$\alpha_{52} = -\frac{\lambda_0}{hk}\left(Z_3(1)\sin\frac{\theta}{2} - Z_6(1)\cos\frac{\theta}{2}\right) + n_1\frac{\lambda_0^2}{k}[Z_4(1) - Z_5(1)] - \frac{\mu}{\beta k}n_2[Z_5(1)H - \lambda_0^2 F Z_4(1)]$$

$$\alpha_{53} = \frac{\lambda_0}{k}\left(Z_4(1)\cos\frac{\theta}{2} + Z_5(1)\sin\frac{\theta}{2}\right) - n_1\frac{\lambda_0^2}{k}[Z_3(1)\cos\theta - Z_6(1)\sin\theta] + \frac{\mu}{\beta k}n_2[Z_3(1)H - Z_6(1)F]$$

$$\beta_{51} = \frac{\lambda_1}{k}\cosh\lambda_1 - n_1\frac{\lambda_1^2}{k}\sinh\lambda_1 - \frac{\mu}{\beta k}n_2 S\sinh\lambda_1$$

$$\beta_{52} = -\frac{\lambda_0}{k}\left(Z_4(1)\sin\frac{\theta}{2} - Z_5(1)\cos\frac{\theta}{2}\right) + n_1\frac{\lambda_0^2}{k}[Z_3(1)\sin\theta - Z_6(1)\cos\theta] - \frac{\mu}{\beta k}n_2[Z_6(1)H - \lambda_0^2 Z_3(1)F]$$

$$\beta_{53} = \frac{\lambda_0}{k}\left(Z_3(1)\cos\frac{\theta}{2} + Z_6(1)\sin\frac{\theta}{2}\right) - n_1\frac{\lambda_0^2}{k}[Z_4(1)\cos\theta + Z_5(1)\sin\theta] + \frac{\mu}{\beta k^2}n_2[Z_4(1)H - Z_5(1)F]$$

$$\alpha_{61} = \frac{\mu}{\beta k^2}(H\cosh\lambda_1 - n_3 S\sinh\lambda_1)$$

$$\alpha_{62} = \frac{\mu}{\beta k^2}[Z_5(1)H - \lambda_0^2 Z_4(1)F] - n_3\frac{\mu}{\beta k^2}\left[\lambda_0\cos\frac{\theta}{2}HZ_5(1) - {} - \lambda_0\sin\frac{\theta}{2}HZ_3(1) - \lambda_0^3\left(\cos\frac{\theta}{2}Z_3(1)F + \sin\frac{\theta}{2}Z_6(1)F\right)\right]$$

$$\alpha_{63} = -\frac{\mu}{\beta k^2}[Z_3(1)H - Z_6(1)F] - n_3\frac{\mu}{\beta k^2}\left[-\lambda_0\cos\frac{\theta}{2}Z_4(1)H - {} - \lambda_0\sin\frac{\theta}{2}Z_5(1)H + \lambda_0^3\cos\frac{\theta}{2}Z_5(1)F - \lambda_0^3\sin\frac{\theta}{2}Z_4(1)F\right]$$

$$\beta_{61} = \frac{\mu}{\beta k^2}(S\sinh\lambda_1 - n_3\lambda_1 S\cosh\lambda_1)$$

$$\beta_{62} = \frac{\mu}{\beta k^2}[Z_6(1)H - \lambda_0^2 Z_3(1)F] - n_3\frac{\mu}{\beta k^2}\left[\lambda_0\cos\frac{\theta}{2}Z_5(1)H - {} - \lambda_0\sin\frac{\theta}{2}Z_4(1)H - \lambda_0^3\left(\cos\frac{\theta}{2}Z_4(1)F + \sin\frac{\theta}{2}Z_5(1)F\right)\right]$$

$$\beta_{63} = \frac{\mu}{\beta k^2}[-Z_4(1)H + Z_5(1)F] - n_3\frac{\mu}{\beta k^2}\left[-\lambda_0\cos\frac{\theta}{2}Z_3(1)H - {} - \lambda_0\sin\frac{\theta}{2}Z_6(1)H + \lambda_0^3\cos\frac{\theta}{2}Z_6(1)F - \lambda_0^3\sin\frac{\theta}{2}Z_3(1)F\right]$$

## SOME PROBLEMS OF FREE CONVECTION IN A MACROCAPILLARY

**Abstract**—Solution is given to a number of problems of free convection in incompressible viscous fluid in elementary macrocapillaries with non-uniform temperature distribution at the boundary. The fluid flow structure and effect of a magnetic field on convection in the case of conducting fluid has been studied in detail. The influence of macrocapillary properties on the flow structure, rate of convection and temperature distribution has been estimated.

## QUELQUES PROBLÈMES DE CONVECTION NATURELLE DANS UN MACROCAPILLAIRE

**Résumé**—Une solution est donnée à quelques problèmes de convection naturelle pour un fluide visqueux incompressible dans des macrocapillaires élémentaires, avec une distribution de température non uniforme à la frontière. On a étudié en détail la structure de l'écoulement du fluide et l'effet du champ magnétique sur la convection dans le cas d'une fluide conducteur. On a estimé l'influence des propriétés macrocapillaires sur la structure de l'écoulement, sur la distribution des taux de convection et de température.

## PROBLEME DER FREIEN KONVEKTION IN MAKROKAPILLAREN

**Zusammenfassung**—Es werden Lösungen angegeben für eine Reihe von Problemen bei freier Konvektion eines inkompressiblen, zähen Fluids in elementaren Makrokapillaren mit ungleichförmiger Temperaturverteilung am Rand. Die Struktur der Strömung und der Effekt eines Magnetfeldes auf die Konvektion für den Fall eines leitenden Fluids wurden genau untersucht. Der Einfluss der Eigenschaften der Makrokapillare auf die Struktur der Strömung, der Anteil der Konvektion und die Temperaturverteilung wurden abgeschätzt.